Alfven solitons

1990 J. Phys. A: Math. Gen. 23 439

(http://iopscience.iop.org/0305-4470/23/4/014)

J. Phys. A: Math. Gen. 23 (1990) 439-453. Printed in the UK

Alfven solitons

Nian-Ning Huang† and Zong-Yun Chen‡

† Department of Physics, Wuhan University, Wuhan 430072, Hubei, People's Republic of China

‡ Department of Physics, Huazhong University of Science and Technology, Wuhan 430074, Hubei, People's Republic of China

Received 24 August 1989

Abstract. A method based on the Darboux transformation in the form of pole-expansion is presented for finding soliton solutions of the derivative nonlinear Schrödinger equation. Related matters concerning Alfven waves in plasmas are also discussed. A generalised Zakharov-Shabat system of equations and its reduced form are deduced simply from the Darboux transformations. The expected asymptotic behaviour of the multisoliton solution is derived from its expression in terms of determinants. An explicit expression for the N-soliton solution is given by means of algebraic techniques from the generalised Zakharov-Shabat equations.

1. Introduction

In recent years there has been great interest in nonlinear Alfven waves. Mikhailovskii *et al* (1976) presented arguments for the existence of Alfven solitons. Mio *et al* (1976) obtained a derivative nonlinear Schrödinger (DNLS) equation for Alfven waves in a plasma. Mjølhus (1976) considered the stability and characteristics of large-amplitude waves.

The DNLS equation was shown to be completely integrable and to have the infinity of conservation laws (Kaup and Newell 1978, Wadati *et al* 1979). The DNLS equation has been solved by an appropriate inverse scattering method (Kaup and Newell 1978). Mjølhus and Wyller (1986) discussed processes of soliton formation as well as the effects of finite temperature and conductivity. Based on the Kaup-Newell inverse scattering tranformation scheme, Ichikawa and Abe (1988) analysed the initial value problem of the DNLS equation.

Kaup and Newell (1978) showed that poles of a Jost solution for the DNLS equation must be in pairs located symmetrically about the origin in the complex plane of the spectral parameter. It becomes very complex when one demonstrates the required analyticity of the Jost solution and calculates the expressions for the soliton solutions.

The purposes of the present paper are to give a simple method for finding soliton solutions of the DNLS equation and to discuss related matters concerning Alfven waves. In section 2, we present a method based on the Darboux transformation in the form of a pole expansion. Recently, the same method has been proposed for finding soliton solutions of the nonlinear Schrödinger (NLS) equation (Chen *et al* 1988). Since in this method of using the Darboux transformation to find soliton solutions it is unnecessary to derive either the analyticities of the Jost solutions or the time dependence of the scattering dates, its virtue is more conspicuous for solving the DNLS equation. In the

0305-4470/90/040439+15$03.50

case of the NLS equation, the Darboux transformations can be determined by simply solving differential equations resulting from the Lax equations. This procedure must be modified in the case of the DNLS equation, because the corresponding resulting differential equations are hard to solve. We shall show that the Darboux transformation in the case of the DNLS equation can be specified by the conditions that the inverse Darboux transformation exists and poles of the Darboux transformation and of its inverse are regular points of the Lax pair.

In section 3, a generalised Zakharov-Shabat system of equations and its reduced form are deduced from the determined Darboux transformations. In section 4, to justify the method, the Jost solutions obtained are shown to satisfy the Lax equations of the DNLS equation. In section 5, the N-soliton solution of the DNLS equation is expressed in terms of determinants of the known quantities by solving the reduced Zakharov-Shabat equations. In section 6, the expected asymptotic behaviour of the N-soliton solution is obtained by its expression in terms of determinants. In section 7, the pure algebraic techniques for calculating explicit solutions from the generalised Zakharov-Shabat equations are given. In section 8, the explicit expression for the N-soliton solution of the DNLS equation is written in a form similar to that used in the direct method of Hirota for the NLS equation (Hirota 1973). In section 9, the concluding remarks are given for the initial value problem of the DNLS equation.

2. Darboux transformation in the form of pole expansion

We shall consider the DNLS equation

$$\mathrm{i}u_t + u_{xx} + \mathrm{i}(|u|^2 u)_x = 0 \tag{1}$$

whose Lax equations are

$$\partial_x F(xt\zeta) = L(xt\zeta)F(xt)\zeta) \tag{2}$$

$$\partial_t F(xt\zeta) = M(xt\zeta)F(xt\zeta) \tag{3}$$

where

$$L(\zeta) = -\mathrm{i}\zeta^{-2}\sigma_3 + \zeta^{-1}U \tag{4}$$

$$m(\zeta) = -\mathrm{i}2\zeta^{-4}\sigma_3 + 2\zeta^{-3}U - \mathrm{i}\zeta^{-2}U^2\sigma_3 - \zeta^{-1}(\mathrm{i}U_x\sigma_3 - U^3) \tag{5}$$

$$U(xt) = \begin{pmatrix} 0 & u(xt) \\ -\overline{u(xt)} & 0 \end{pmatrix} \tag{6}$$

and the overbar denotes the complex conjugate. The compatibility condition of (2) and (3) yields

$$U_t + \mathrm{i}U_{xx}\sigma_3 - U_x^3 = 0 \tag{7}$$

whose matrix elements are (1) and its complex conjugate.

$U_0 = 0$ is obviously a solution of (7), its related Jost solution of (2) and (3) is

$$F_0(zt\zeta) = \exp[-\mathrm{i}(\zeta^{-2}x + 2\zeta^{-4}t)\sigma_3]. \tag{8}$$

Since

$$L(\zeta) \to 0 \qquad M(\zeta) \to 0 \qquad \text{as } |\zeta| \to \infty \tag{9}$$

we may demand that

$$F(\zeta)\to I \qquad \text{as } |\zeta|\to\infty. \tag{10}$$

We try to find the Jost solution of the form

$$F(xt\zeta)=G(xt\zeta)F_0(xt\zeta) \tag{11}$$

$$G(\zeta)\to I \qquad \text{as } |\zeta|\to\infty. \tag{12}$$

From (4)-(6), we have

$$\sigma_3 L(-\zeta)\sigma_3=L(\zeta) \qquad \sigma_3 M(-\zeta)\sigma_3=M(\zeta) \tag{13}$$

and then

$$\sigma_3 F(-\zeta)\sigma_3=F(\zeta) \qquad \sigma_3 G(-\zeta)\sigma_3=G(\zeta). \tag{14}$$

We propose an ansatz that $G(\zeta)$ is meromorphic and, further, has only poles of order one. The right-hand equation of (14) shows that poles of $G(\zeta)$ must be in pairs located symmetrically about the origin of the complex ζ plane. In the case of $2N$ poles, by virtue of (12) and (14), we have

$$G(xt\zeta)=I+\sum_{j=1}^{N}\frac{1}{\zeta-\zeta_j}A_j-\sum_{j=1}^{N}\frac{1}{\zeta+\zeta_j}\sigma_3A_j\sigma_3 \tag{15}$$

where ζ_j is a complex constant and A_j is the residue of $G(xt\zeta)$ at ζ_j.

From (4)-(6), we have

$$-L^\dagger(\bar\zeta)=L(\zeta) \qquad -M^\dagger(\bar\zeta)=M(\zeta) \tag{16}$$

and then

$$F^{-1}(\zeta)=F^\dagger(\bar\zeta) \qquad G^{-1}(\zeta)=G^\dagger(\bar\zeta) \tag{17}$$

where the dagger means the Hermitian conjugate. Thus we have also

$$G^{-1}(xt\zeta)=I+\sum_{k=1}^{N}\frac{1}{\zeta-\bar\zeta_k}A_k^\dagger-\sum_{k=1}^{N}\frac{1}{\zeta+\bar\zeta_k}\sigma_3A_k^\dagger\sigma_3. \tag{18}$$

We now consider a series of transformations, each of which has a couple of poles,

$$D_j(xt\zeta)=I+\frac{1}{\zeta-\zeta_j}B_j-\frac{1}{\zeta+\zeta_j}\sigma_3B_j\sigma_3 \tag{19}$$

$$D_j^{-1}(xt\zeta)=I+\frac{1}{\zeta-\bar\zeta_j}B_j^\dagger-\frac{1}{\zeta+\bar\zeta_j}\sigma_3B_j^\dagger\sigma_3 \tag{20}$$

$$F_j(xt\zeta)=D_j(xt\zeta)F_{j-1}(xt\zeta) \tag{21}$$

$$F_j^{-1}(xt\zeta)=F_{j-1}^{-1}(xt\zeta)D_j^{-1}(xt\zeta) \tag{22}$$

$$G(xt\zeta)=D_N(xt\zeta)D_{N-1}(xt\zeta)\dots D_1(xt\zeta) \tag{23}$$

$$G^{-1}(xt\zeta)=D_1^{-1}(xt\zeta)D_2^{-1}(xt\zeta)\dots D_N^{-1}(xt\zeta). \tag{24}$$

The transformation $D_j(xt\zeta)$ is referred to as the Darboux transformation, we write it in the pole-expansion form instead of its usual power series expansion form.

Taking the limit as $\zeta\to\zeta_j$, from $D_jD_j^{-1}=D_j^{-1}D_j=I$, we have

$$B_jD_j^{-1}(\zeta_j)=D_j^{-1}(\zeta_j)B_j=0. \tag{25}$$

This yields that B_j is degenerate, and may be written as

$$B_j = \begin{pmatrix} g_j' \\ h_j' \end{pmatrix} (g_j \, h_j) \tag{26}$$

where

$$g_j' = \frac{\bar{g}_j}{2} \frac{\zeta_j^2 - \bar{\zeta}_j^2}{\zeta_j |g_j|^2 + \bar{\zeta}_j |h_j|^2} \tag{27}$$

$$h_j' = \frac{\bar{h}_j}{2} \frac{\zeta_j^2 - \bar{\zeta}_j^2}{\bar{\zeta}_k |g_j|^2 + \zeta_j |h_j|^2} \tag{28}$$

For the Jost solution $F_j(xt\zeta)$, (2) and (3) are

$$\partial_x F_j(xt\zeta) = L_j(xt\zeta) F_j(xt\zeta) \tag{29}$$

$$\partial_t F_j(xt\zeta) = M_j(xt\zeta) F_j(xt\zeta) \tag{30}$$

where L_j and M_j are obtained from (4)-(6) by setting a particular U_j, which we shall determine. From (29), we have

$$\partial_x F_j(xt\zeta) F_j^{-1}(xt\zeta) = L_j(xt\zeta). \tag{31}$$

Taking the limit as $\zeta \to \zeta_j$, we have

$$\{\partial_x [B_j F_{j-1}(xt\zeta_j)]\} F_{j-1}^{-1}(xt\zeta_j) D_j^{-1}(xt\zeta_j) = 0 \tag{32}$$

if ζ_j is a regular point of L_j. By virtue of (25), we have

$$\sigma_x [B_j F_{j-1}(xt\zeta_j)] = \ldots B_j F_{j-1}(xt\zeta_j) \tag{33}$$

where the expression on the left of B_j in the right-hand side is not obviously written. Similarly, we have

$$\partial_t [B_j F_{j-1}(xt\zeta_j)] = \ldots B_j F_{j-1}(xt\zeta_j). \tag{34}$$

Substituting (26) into (33) and (34), we find

$$(g_j h_j) = (b_j c_j) F_{j-1}^{-1}(xt\zeta_j) \tag{35}$$

where b_j and c_j are constants. B_j is thus completely specified.

3. The generalised Zakharov–Shabat equations

From (27) and (28), we have

$$\sigma_2 B_j^{\mathrm{T}} \sigma_2 = \frac{\zeta_j^2 - \bar{\zeta}_j^2}{2\zeta_j} D_j^{-1}(\zeta_j) \tag{36}$$

$$\sigma_2 D_j(\zeta)^{\mathrm{T}} \sigma_2 = \frac{\zeta^2 - \bar{\zeta}_j^2}{\zeta^2 - \zeta_j^2} D_j^{-1}(\zeta) \tag{37}$$

where the superscript T means the transpose. From (23) and (15), we have

$$A_j = \lim_{\zeta \to \zeta_j} (\zeta - \zeta_j) G(\zeta)$$

$$= D_N(\zeta_j) \ldots D_{j+1}(\zeta_j) B_j D_{j-1} \ldots D_1(\zeta_j). \tag{38}$$

By virtue of (36) and (37), we have

$$A_j = a_j^{-1}\sigma_2 G^{-1}(\zeta_j)^{\mathrm{T}}\sigma_2 \tag{39}$$

where

$$a_j = \prod_{k(\neq j)} \frac{\zeta_j^2-\zeta_k^2}{\zeta_j^2-\bar{\zeta}_k^2}\,\frac{2\zeta_j}{\zeta_j^2-\bar{\zeta}_j^2}. \tag{40}$$

Therefore, we may write (15) as

$$G(\zeta) = I + \sum_{j=1}^{N} \frac{1}{\zeta-\zeta_j}\, a_j^{-1}\sigma_2 g^{-1}(\zeta_j)^{\mathrm{T}}\sigma_2 - \sum_{j=1}^{N}\frac{1}{\zeta+\zeta_j}\, a_j^{-1}\sigma_3\sigma_2 G^{-1}(\zeta_j)^{\mathrm{T}}\sigma_2\sigma_3. \tag{41}$$

From (26), (35), (38) and (39), we may write

$$\sigma_2 G^{-1}(\zeta_j)^{T}\sigma_2 = \begin{pmatrix}\varphi_j\\ \psi_j\end{pmatrix}(b_j\, c_j)F_0^{-1}(\zeta_j) \tag{42}$$

and then

$$G(\bar{\zeta}_k) = \begin{pmatrix}\bar{\psi}_k\\ -\bar{\varphi}_k\end{pmatrix}(\bar{c}_k - \bar{b}_k)F_0^{-1}(\bar{\zeta}_k) \tag{43}$$

on account of (17). Although φ_j, etc can be expressed in terms of the known quantities, they may be directly determined in the following manner.

Setting $\zeta=\bar{\zeta}_k$ in (41), and taking account of (42) and (43), we obtain

$$\begin{pmatrix}\bar{\psi}_k\\ -\bar{\varphi}_k\end{pmatrix}(\bar{f}_k^{-1}-\bar{f}_k)$$
$$= I + \sum_{j=1}^{N}\frac{1}{\bar{\zeta}_k-\zeta_j}\, a_j^{-1}\begin{pmatrix}\varphi_j\\ \psi_j\end{pmatrix}(f_j f_j^{-1}) - \sum_{j=1}^{N}\frac{1}{\bar{\zeta}_k+\zeta_j}\, a_j^{-1}\begin{pmatrix}\varphi_j\\ -\psi_j\end{pmatrix}(f_j - f_j^{-1}). \tag{44}$$

Here we have taken $c_j = b_j^{-1}$ without loss of generality, and have written

$$f_j = b_j \exp \mathrm{i}(\zeta_j^{-2} + 2\zeta_j^{-4}t). \tag{45}$$

Equations (44) are obviously referred to as the generalised Zakharov-Shabat equations; the N-soliton solution of the DNLS equation can be determined from these $2N$ linear algebraic equations.

Multiplying (44) by $(\bar{f}_k\bar{f}_k^{-1})^{\mathrm{T}}$ from right, we obtain

$$\begin{pmatrix}\bar{f}_l\\ \bar{f}_k^{-1}\end{pmatrix} + \sum_{j=1}^{N}\frac{1}{\bar{\zeta}_k-\zeta_j}\, a_j^{-1}\begin{pmatrix}\varphi_j\\ \psi_j\end{pmatrix}(f_j\bar{f}_k + f_j^{-1}\bar{f}_k^{-1})$$
$$-\sum_{j=1}^{N}\frac{1}{\bar{\zeta}_k+\zeta_j}\, a_j^{-1}\begin{pmatrix}\varphi_j\\ -\psi_j\end{pmatrix}(f_j\bar{f}_k - f_j^{-1}\bar{f}_k^{-1}) = 0. \tag{46}$$

Equation (46) is obviously the reduced form of the generalised Zakharov-Shabat equations, since φ_j and ψ_j can both be determined by solving these N linear algebraic equations in the N-soliton case.

4. Demonstration

We ought to show that the Jost solution obtained by the above procedure satisfies the corresponding Lax equations. Equation (35) ensures that $\pm\zeta_j$ $(j = 1, 2, \ldots, N)$ and their complex conjugates are regular points of $[\partial_x F(xt\zeta)]F^{-1}(xt\zeta)$. From (11), we have

$$[\partial_x F(xt\zeta)]F^{-1}(xt\zeta) = G_x(\zeta)G^{-1}(\zeta) - \mathrm{i}\zeta^{-2}G(\zeta)\sigma_3 G^{-1}(\zeta). \tag{47}$$

The factor $[\partial_x F(xt)\zeta)]F^{-1}(xt\zeta)$ is thus analytic everywhere except at $\zeta = 0$.

We expand $G(\zeta)$ into a Taylor series about $\zeta = 0$:

$$G(\zeta) = \sum_{n=0}^{\infty} \rho_n \zeta^n \tag{48}$$

and then

$$G^{-1}(\zeta) = \sum_{m=0}^{\infty} \rho_m^{\dagger} \zeta^m \tag{49}$$

where

$$\sum_{n+m=l} \rho_n \rho_m^{\dagger} = \delta_{l0} I \tag{50}$$

$$\rho_n = \delta_{n0} I - \sum_{J=1}^{N} \zeta_j^{-n-1}[A_j + (-1)^n \sigma_3 A_j \sigma_3]. \tag{51}$$

Substituting (48) and (49) into (47), we have

$$[\partial_x F(xt\zeta)]F^{-1}(xt\zeta) = \zeta^{-2}Q_{-2} + \zeta^{-1}Q_{-1} + Q_0 + \dots \tag{52}$$

where

$$Q_{-2} = -\mathrm{i}(\rho_0 \sigma_3 \rho_0^{\dagger}) \tag{53}$$

$$Q_{-1} = -\mathrm{i}(\rho_1 \sigma_3 \rho_0^{\dagger} + \rho_0 \sigma_3 \rho_1^{\dagger}). \tag{54}$$

From (51), we can see that ρ_{even} are diagonal matrices that commute with σ_3, and ρ_{odd} are matrices with vanishing diagonal elements that anticommute with σ_3. We thus have

$$-\mathrm{i}(\rho_0 \sigma_3 \rho_0^{\dagger}) = -\mathrm{i}\sigma_3 \tag{55}$$

$$-\mathrm{i}(\rho_1 \sigma_3 \rho_0^{\dagger} + \rho_0 \sigma_3 \rho_1^{\dagger}) = -2\mathrm{i}(\rho_1 \rho_0^{\dagger} \sigma_3) = U \tag{56}$$

due to (50), and since $\rho_1 \rho_0^{\dagger}$ is a matrix with vanishing diagonal elements satisfying

$$-2\mathrm{i}(\rho_1 \rho_0^{\dagger} \sigma_3) = -[-2\mathrm{i}(\rho_1 \rho_0^{\dagger} \sigma_3)]^{\dagger} \tag{57}$$

we can identify it with U.

Therefore, $[\partial_x F(xt\zeta)]F^{-1}(xt\zeta) - (-\mathrm{i}\zeta^{-2}\sigma_3 + \zeta^{-1}U)$ is analytic in the whole complex ζ plane and tends to zero as $|\zeta| \to \infty$, (10), by the Liouville theorem, it is equal to zero. This yields

$$G_x(\zeta)G^{-1}(\zeta) - \mathrm{i}\gamma^{-2}G(\zeta)\sigma_3 G^{-1}(\zeta) = -\mathrm{i}\zeta^{-2}\sigma_3 + \zeta^{-1}U \tag{58}$$

or (2).

From (2), we have

$$\partial_x^2 F(xt\zeta) = [L^2(xt\zeta) + L_x(xt\zeta)]F(xt\zeta) \tag{59}$$

and then

$$G_{xx}(\zeta) - 2\mathrm{i}\zeta^{-2}G_x(\zeta)\sigma_3 = \zeta^{-2}U^2 G(\zeta) + \zeta^{-1}U_x G(\zeta). \tag{60}$$

Multiplying it by $\sigma_3 G^{-1}(\zeta)$ from the right, we have

$$G_{xx}(\zeta)\sigma_3 g^{-1}(\zeta) - 2\mathrm{i}\zeta^{-2}G_x(\zeta)G^{-1}(\zeta) = (\zeta^{-2}U^2 + \zeta^{-1}U_x)G(\zeta)\sigma_3 G^{-1}(\zeta). \tag{61}$$

From (58) and (61), we have

$$-\mathrm{i}(\rho_2\sigma_3\rho_0^{+}+\ldots+\rho_0\sigma_3\rho_2^{\dagger})=-\tfrac{1}{2}\mathrm{i}\,U^2\sigma_3 \tag{62}$$

$$-\mathrm{i}(\rho_3\sigma_3\rho_0^{\dagger}+\ldots+\rho_0\sigma_3\rho_3^{\dagger})=\tfrac{1}{2}U^3-\tfrac{1}{2}\mathrm{i}\,U_x\sigma_3. \tag{63}$$

From (11), we have

$$[\sigma_t F(xt\zeta)]F^{-1}(xt\zeta)=G_t(\zeta)G^{-1}(\xi)-2\mathrm{i}\zeta^{-4}G(\zeta)\sigma_3G^{-1}(\zeta). \tag{64}$$

Equation (35) ensures that (64) is analytic everywhere except at $\zeta=0$. Substituting (48) and (49) into (64), and taking account of (55), (56), (62) and (63), we can see that the principal part of (64) at $\zeta=0$ is just $M(xt\zeta)$ in (5). Therefore, $[\partial_t F(xt\zeta)]F^{-1}(xt\zeta)-M(xt\zeta)$ is analytic in the whole complex ζ plane and tends to zero, (10), by Liouville theorem it is equal to zero. This yields (3). We have shown that the Jost solution obtained by the above procedure satisfies the corresponding Lax equations. From (56), the N-soliton solution of the DNLS equation can be given by

$$u_N=2\mathrm{i}(\rho_1)_{12}\overline{(\rho_0)_{22}}. \tag{65}$$

5. Expressions of multisoliton solutions in terms of determinants of the known quantities

From (46), we have

$$2\sum_{j=1}^{N} a_j^{-1}\varphi_j f_j^{-1}K_{jk}=\bar f_k^2 \tag{66}$$

$$2\sum_{j=1}^{N} a_j^{-1}\psi_j f_j^{-1}(K^{\dagger})_{jk}=-1 \tag{67}$$

where

$$K_{jk}=\frac{1}{\zeta_j^2-\bar\zeta_k^2}(\zeta_j f_j^2\bar f_k^2+\bar\zeta_k). \tag{68}$$

Substituting (51) into (65), we have

$$(\rho_1)_{12}=-2\sum_j a_j^{-1}\varphi_j f_j^{-1}\zeta_j^{-2} \tag{69}$$

$$(\rho_0)_{22}=1-2\sum_j a_j^{-1}\psi_j f_j^{-1}\zeta_j^{-1} \tag{70}$$

on account of (42). By virtue of (66) and (67), we obtain

$$(\rho_1)_{12}=-\sum_{j,k}\bar f_k^2(K^{-1})_{kj}\zeta_j^{-2} \tag{71}$$

$$(\rho_0)_{22}=1+\sum_{j,k}[(K^{\dagger})^{-1}]_{kj}\zeta_j^{-1} \tag{72}$$

and then

$$\overline{(\rho_0)_{22}}=1+\sum_{j,k}\bar\zeta_k^{-1}(K^{-1})_{kj}. \tag{73}$$

Since we have the known linear algebra formula,

$$\det(p_iq_j+R_{ij})=\det R\left(1+\sum_{i,j=1}^{M}p_iq_j(R^{-1})_{ji}\right) \tag{74}$$

which is valid for a non-singular $M\times M$ matrix R and arbitrary rows p and q, (71) and (73) can be rewritten as

$$(\rho_1)_{12}=-[(\det K)^{-1}\det K'-1] \tag{75}$$

$$\overline{(\rho_0)_{22}}=(\det K)^{-1}\det K'' \tag{76}$$

where

$$K'_{jk}=K_{jk}+\zeta_j^{-2}\bar{f}_k^2 \tag{77}$$

$$K''_{jk}=K_{jk}+\bar{\zeta}_k^{-1}=\frac{1}{\zeta_j^2-\bar{\zeta}_k^2}\frac{\zeta_j}{\bar{\zeta}_k}(\bar{\zeta}_k f_j^2\bar{f}_k^2+\zeta_j). \tag{78}$$

Substituting (75) and (76) into (65), the N-soliton solution of the DNLS equation has been expressed in terms of determinants of the known quantities.

From (45), we may write

$$f_j=\exp[\mathrm{i}r_j-s_j] \tag{79}$$

where

$$r_j=(\mathrm{Re}\,\zeta_j^{-2})x+2[(\mathrm{Re}\,\zeta_j^{-2})^2-(\mathrm{Im}\,\alpha_j^{-2})^2]t+\delta_j \tag{80}$$

$$s_j=(\mathrm{Im}\,\zeta_j^{-2})[x-x_j+4(\mathrm{Re}\,\zeta_j^{-2})t] \tag{81}$$

$$b_j=\exp[\mathrm{i}\delta_j+(\mathrm{Im}\,\zeta_j^{-2})x_j]. \tag{82}$$

From (75), (76) and (65), we easily find the one-soliton solution of the DNLS equation:

$$\begin{aligned}u_1&=2\mathrm{i}\frac{\zeta_1^2-\bar{\zeta}_1^2}{\zeta_1^2}\frac{\bar{f}_1^2}{\zeta_1|f_1|^4+\bar{\zeta}_1}\frac{\bar{\zeta}_1|f_1|^4+\zeta_1}{\zeta_1|f_1|^4+\bar{\zeta}_1}\\&=2\mathrm{i}(\bar{\lambda}_1-\lambda_1)\frac{\exp-\mathrm{i}2r_1}{\cosh 2s_1}\left(1-\frac{\lambda_1-\bar{\lambda}_1}{\lambda_1+\bar{\lambda}_1}\tanh 2s_1\right)\\&\quad\times\left(1+\frac{\lambda_1-\bar{\lambda}_1}{\lambda_1+\bar{\lambda}_1}\tanh 2s_1\right)^{-2}\end{aligned} \tag{83}$$

where

$$\zeta_j^{-1}=\lambda_j=\lambda'_j+\mathrm{i}\lambda''_j. \tag{84}$$

Equation (83) can be written as

$$u_1=g_1\exp-\mathrm{i}\phi_1 \tag{85}$$

where

$$\phi_1=2r_1+3\tan^{-1}\left(\frac{\lambda''_1}{\lambda'_1}\tanh 2s_1\right) \tag{86}$$

$$g_1^2=(4\lambda''_1)^2\left[\left(1+\frac{\lambda_1''^2}{\lambda_1'^2}\right)\cosh^2 2s_1-\frac{\lambda_1''^2}{\lambda_1'^2}\right]^{-1} \tag{87}$$

or

$$g_1^2=2(4\lambda'_1\lambda''_1)^2\left[(\lambda_1'^2+\lambda_1''^2)\cosh 4s_1+(\lambda_1'^2-\lambda_1''^2)\right]^{-1}. \tag{88}$$

Equation (83) is the same as that given by Kaup and Newell (1978); equations (88) and (87) are closely related to the expression of Mjølhus (1978) and that of Anderson and Lisak (1983), respectively.

6. Asymptotic behaviours of multisoliton solutions

The expected asymptotic behaviour of the N-soliton solution will be obtained directly from its expression in terms of determinants, (65), (75) and (76). The calculation is performed for the case of positive $\mathrm{Im}\,\zeta_j^{-2}$, $j=1, 2, \ldots, N$, it can be extended to the general cases without difficulty. We also assume that

$$(\mathrm{Re}\,\zeta_1^{-2}) > (\mathrm{Re}\,\zeta_2^{-2}) > \ldots > (\mathrm{Re}\,\zeta_N^{-2}) \tag{89}$$

without loss of generality. The vicinity of $x = x_j - 4(\mathrm{Re}\,\zeta_j^{-2})t$ is denoted by Ω_j. In the limit as $t \to \infty$, these vicinities must be separated from left to right as

$$\Omega_N, \Omega_{N-1}, \ldots, \Omega_1. \tag{90}$$

In the vicinity Ω_m,

$$(x - x_j) + 4(\mathrm{Re}\,\zeta_j^{-2})t \to \infty \qquad |f_j| \to 0 \qquad j < m \tag{91}$$

$$(x - x_k) + 4(\mathrm{Re}\,\zeta_k^{-2})t \to -\infty \qquad |f_k| \to \infty \qquad k > m. \tag{92}$$

det K then approaches det K_∞:

$$\det K_\infty = \begin{vmatrix} \dfrac{\bar\zeta_j}{\zeta_i^2 - \bar\zeta_j^2} & \dfrac{\bar\zeta_m}{\zeta_i^2 - \bar\zeta_m^2} & 0 \\ \dfrac{\bar\zeta_j}{\zeta_m^2 - \bar\zeta_j^2} & \dfrac{\bar\zeta_m}{\zeta_m^2 - \bar\zeta_m^2} + \dfrac{\zeta_m |f_m|^4}{\zeta_m^2 - \bar\zeta_m^2} & \dfrac{\zeta_m f_m^2 \bar f_l^2}{\zeta_m^2 - \bar\zeta_l^2} \\ 0 & \dfrac{\zeta_k f_k^2 \bar f_m^2}{\zeta_k^2 - \bar\zeta_m^2} & \dfrac{\zeta_k f_k^2 \bar f_l^2}{\zeta_k^2 - \bar\zeta_l^2} \end{vmatrix}. \tag{93}$$

Hereafter i and j run from 1 to $m-1$, and k, l run from $m+1$ to N. In (93), we reserve those elements which contribute to the determinant terms of the order of $|f_{m+1}|^4 |f_{m+2}|^4 \ldots |f_N|^4$.

The RHS of (93) can obviously be decomposed into two determinants, each of which is proportional to the determinant D_m

$$D_m = \begin{vmatrix} \dfrac{\bar\zeta_j}{\zeta_i^2 - \bar\zeta_j^2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \dfrac{\zeta_k f_k^2 \bar f_l^2}{\zeta_k^2 - \bar\zeta_l^2} \end{vmatrix} \tag{94}$$

so that

$$\det K_\infty = \frac{1}{\zeta_m^2 - \bar\zeta_m^2} (\bar\zeta_m |\alpha(\zeta_m)|^2 + \zeta_m |\beta(\zeta_m)|^2 |f_m|^4) D_m \tag{95}$$

where

$$\alpha(\zeta_m) = \prod_{j=1}^{m-1} \frac{\zeta_m^2 - \zeta_j^2}{\zeta_m^2 - \bar\zeta_j^2} \tag{96}$$

$$\beta(\zeta_m) = \prod_{k=m+1}^{N} \frac{\zeta_m^2 - \zeta_k^2}{\zeta_m^2 - \bar\zeta_k^2}. \tag{97}$$

When $t\to\infty$, in the vicinity Ω_m, det K' approaches det K'_∞

$$\det K_\infty=\begin{vmatrix} \dfrac{\bar\zeta_j}{\zeta_i^2-\bar\zeta_j^2} & \dfrac{\bar\zeta_m}{\zeta_i^2-\bar\zeta_m^2}+\dfrac{\bar f_m^2}{\zeta_i^2} & \dfrac{\bar f_l^2}{\zeta_i^2} \\ \dfrac{\bar\zeta_j}{\zeta_m^2-\bar\zeta_j^2} & \dfrac{\bar\zeta_m}{\zeta_m^2-\bar\zeta_m^2}+\dfrac{\zeta_m|f_m|^4}{\zeta_m^2-\bar\zeta_m^2}+\dfrac{\bar f_m^2}{\zeta_m^2} & \dfrac{\zeta_m f_m^2\bar f_l^2}{\zeta_m^2-\bar\zeta_l^2}+\dfrac{\bar f_l^2}{\zeta_m^2} \\ 0 & \dfrac{\zeta_k f_k^2\bar f_m^2}{\zeta_k^2-\bar\zeta_m^2} & \dfrac{\zeta_k f_k^2\bar f_l^2}{\zeta_k^2-\bar\zeta_l^2} \end{vmatrix}. \tag{98}$$

We decompose it to obtain

$$\det K'_\infty-\det K_\infty=\begin{vmatrix} \dfrac{\bar\zeta_j}{\zeta_i^2-\bar\zeta_j^2} & \dfrac{\bar f_m^2}{\zeta_i^2} & \dfrac{\bar f_l^2}{\zeta_i^2} \\ \dfrac{\bar\zeta_j}{\zeta_m^2-\bar\zeta_j^2} & \dfrac{\bar f_m^2}{\zeta_m^2} & \dfrac{\bar f_l^2}{\zeta_m^2} \\ 0 & \dfrac{\zeta_k f_k^2\bar f_m^2}{\zeta_k^2-\bar\zeta_m^2} & \dfrac{\zeta_k f_k^2\bar f_l^2}{\zeta_k^2-\bar\zeta_l^2} \end{vmatrix}. \tag{99}$$

We then obtain

$$\det K'_\infty-\det K_\infty=\prod_{j=1}^{m-1}\left(\frac{\bar\zeta_j}{\zeta_j}\right)^2\frac{1}{\zeta_m^2}\,\alpha(\zeta_m)\overline{\beta(\zeta_m)}\bar f_m^2 D_m. \tag{100}$$

When $t\to\infty$, in the vicinity Ω_m, det K'' approaches det K''_∞:

$$\det K''_\infty=\prod_{j=1}^{m-1}\frac{\zeta_j}{\bar\zeta_j}\prod_{k=m+1}^{N}\frac{\zeta_k}{\bar\zeta_k}\frac{\zeta_m}{\bar\zeta_m}\begin{vmatrix} \dfrac{\zeta_i}{\zeta_i^2-\bar\zeta_j^2} & \dfrac{\zeta_i}{\zeta_i^2-\bar\zeta_m^2} & 0 \\ \dfrac{\zeta_m}{\zeta_m^2-\bar\zeta_j^2} & \dfrac{\zeta_m}{\zeta_m^2-\bar\zeta_m^2}+\dfrac{\bar\zeta_m|f_m|^4}{\zeta_m^2-\bar\zeta_m^2} & \dfrac{\bar\zeta_l f_m^2\bar f_l^2}{\zeta_m^2-\bar\zeta_l^2} \\ 0 & \dfrac{\bar\zeta_m f_k^2\bar f_m^2}{\zeta_k^2-\bar\zeta_m^2} & \dfrac{\bar\zeta_l f_k^2\bar f_l^2}{\zeta_k^2-\bar\zeta_l^2} \end{vmatrix} \tag{101}$$

on account of (78). We then obtain

$$\det K''_\infty=\prod_{j=1}^{m-1}\left(\frac{\zeta_j}{\bar\zeta_j}\right)^2\frac{1}{\zeta_m^2-\bar\zeta_m^2}\frac{\zeta_m}{\bar\zeta_m}(\zeta_m|\alpha(\zeta_m)|^2+\bar\zeta_m|\beta(\zeta_m)|^2|f_m|^4)D_m. \tag{102}$$

Therefore, we obtain

$$(\rho_1)_{12}\approx\prod_{j=1}^{m-1}\left(\frac{\bar\zeta_j}{\zeta_j}\right)^2\frac{\zeta_m^2-\bar\zeta_m^2}{\zeta_m^2}\frac{\bar f_m^{(+)2}}{\bar\zeta_m+\zeta_m|f_m^{(+)}|^4} \tag{103}$$

$$\overline{(\rho_0)_{22}}\approx\prod_{j=1}\left(\frac{\zeta_j}{\bar\zeta_j}\right)^2\frac{\zeta_m}{\bar\zeta_m}\frac{\zeta_m+\bar\zeta_m|{}_m^{(+)}|^4}{\bar\zeta_m+\zeta_m|f_m^{(+)}|^4} \tag{104}$$

$$u_N\approx 2\mathrm{i}\,\frac{\zeta_m^2-\bar\zeta_m^2}{\zeta_m^2}\frac{\zeta_m}{\bar\zeta_m}\bar f_m^{(+)2}\frac{\zeta_m+\bar\zeta_m|f_m^{(+)}|^4}{(\bar\zeta_m+\zeta_m|f_m^{(+)}|^4)^2} \tag{105}$$

where

$$f_m^{(+)2}=f_m^2\beta(\zeta_m)\alpha(\zeta_m)^{-1}. \tag{106}$$

Similarly, when $\to -\infty$, in the vicinity Ω_m, we have

$$u_n \approx 2\mathrm{i}\,\frac{\zeta_m^2-\bar{\zeta}_m^2}{\zeta_m^2}\,\frac{\zeta_m}{\bar{\zeta}_m}\,\bar{f}_m^{(-)2}\,\frac{\zeta_m+\bar{\zeta}_m|f_m^{(-)}|^4}{(\bar{\zeta}_m+\zeta_m|f_m^{(-)}|^4)^2} \tag{107}$$

where

$$f_m^{(-)2}=f_m^2\beta(\zeta_m)^{-1}\alpha(\zeta_m). \tag{108}$$

Comparing (105) and (107) with the single soliton solution related to spectral parameter ζ_m, the additional phases and displacements of peak, $\delta_m^{(\pm)}$ and $\Delta_m^{(\pm)}$, are easily derived:

$$f_m^{(\pm)}=f_m\exp[\mathrm{i}\delta_m^{(\pm)}-(\mathrm{Im}\,\zeta_m^{-2})\Delta_m^{(\pm)}] \tag{109}$$

$$\delta_m^{(\pm)}=\pm\tfrac{1}{2}[\arg\beta(\zeta_m)-\arg\alpha(\zeta_m)] \tag{110}$$

$$\Delta_m^{(\pm)}=\pm\tfrac{1}{2}(\mathrm{Im}\,\zeta_m^{-2})^{-1}[\ln|\alpha(\zeta_m)|-\ln|\beta(\zeta_m)|]. \tag{111}$$

7. Method for explicitly solving the generalised Zakharov–Shabat equations

From (44), we obtain a system of linear algebraic equations

$$-\bar{\psi}_k=\sum_{j=1}^{N}\bar{f}_k^{-1}\frac{2\bar{\zeta}_k}{\bar{\zeta}_k^2-\zeta_j^2}f_j^{-1}a_j^{-1}\varphi_j \tag{112}$$

$$\varphi_k=f_k^{-1}+\sum_{j=1}^{N}f_k^{-1}\frac{2\bar{\zeta}_j}{\zeta_k^2-\bar{\zeta}_j^2}\bar{f}_j^{-1}\bar{a}_j^{-1}\bar{\psi}_j. \tag{113}$$

If we introduce

$$p_j=-\mathrm{i}\zeta_j \tag{114}$$

$$\alpha_j=\tfrac{1}{2}\mathrm{i}a_j=\prod_{k(\neq j)}\frac{p_j^2-p_k^2}{p_j^2-\bar{p}_k^2}\,\frac{p_j}{p_j^2-\bar{p}_j^2} \tag{115}$$

$$C=(\alpha_1^{-1/2}f_1^{-1},\ldots,\alpha_N^{-1/2}f_N^{-1}) \tag{116}$$

$$\Phi=(\alpha_1^{-1/2}\varphi_1,\ldots,\alpha_N^{-1/2}\varphi_N) \tag{117}$$

$$\Psi=(\alpha_1^{-1/2}\psi_1,\ldots,\alpha_N^{-1/2}\psi_N) \tag{118}$$

$$Q_{jk}=C_j\frac{\bar{p}_k}{p_j^2-\bar{p}_k^2}\bar{C}_k \tag{119}$$

then (112) and (113) can be written as

$$\bar{\Psi}=-\Phi Q \tag{120}$$

$$\Phi=C+\bar{\Psi}Q^{\mathrm{T}} \tag{121}$$

and (69) can be expressed as

$$(\rho_1)_{12}=\mathrm{i}C(I+QQ^{\mathrm{T}})^{-1}p^{-2}C^{\mathrm{T}} \tag{122}$$

where p^{-2} is simply a diagonal matrix, i.e. $\mathrm{diag}(p_1^{-2},\ldots,p_N^{-2})$.

By virtue of (74), we have

$$\begin{aligned}(\rho_1)_{12}&=\mathrm{i}\sum_{j,k}C_j[(I+R)^{-1}]_{jk}p_k^{-2}C_k\\&=\mathrm{i}\{[\det(I+R)]^{-1}\det(I+R')-1\}\end{aligned} \tag{123}$$

where

$$R = QQ^{\mathrm{T}} \tag{124}$$

$$R'_{jk} = R_{jk} + p_j^{-2} C_j C_k. \tag{125}$$

It is convenient to express (125) as

$$R' = Q' Q''^{\mathrm{T}} \tag{126}$$

where Q' and Q'' are matrices whose rows are extended from 1 to N, and columns from 0 to N

$$Q'_{jk} = Q_{jk} \qquad Q'_{j0} = p_j^{-2} C_j \tag{127}$$

$$Q''_{jk} = Q_{jk} \qquad Q''_{j0} = C_j. \tag{128}$$

We have

$$\det(I+R) = 1 + \sum_{r=1}^{N} \sum_{1 \leqslant j_1 < j_2 < \ldots < j_r \leqslant N} R(j_1, j_2, \ldots, j_r) \tag{129}$$

and

$$R(j_1, j_2, \ldots, j_r) = \sum_{1 \leqslant k_1 < k_2 < \ldots < k_r \leqslant N} Q(j_1, j_2, \ldots, j_r; k_1, k_1, k_2, \ldots, k_r) \tag{130}$$

by virtue of the Binet-Cauchy formula, where $Q(j_1, j_2, \ldots, j_r; k_1, k_2, \ldots, k_r)$ denotes a minor, which is the determinant of a submatrix of Q consisting of $(j_1, j_2, \ldots, j_r)$ rows and $(k_1, k_2, \ldots, k_r)$ columns. $Q(j_1, j_2, \ldots, j_r)$ means a principal minor, i.e. $Q(j_1, j_2, \ldots, j_r; j_q, j_2, \ldots, j_r)$.

Using the known formula

$$\det\left(\frac{1}{x_j + y_k}\right) = \prod_{j<j'} (x_j - x_{j'}) \prod_{k<k'} (y - y_{k'}) \prod_{j,k} (x_j + y_k)^{-1} \tag{131}$$

we obtain

$$Q(j_1, j_2, \ldots, j_r; k_1, k_2, \ldots, k_r)^2$$
$$= \prod_j C_j^2 \prod_k \bar{p}_k^2 \bar{C}_k^2 \prod_{j<j'} (p_j^2 - p_{j'}^2)^2 \prod_{k<k'} (\bar{p}_k^2 - \bar{p}_{k'}^2)^2 \prod_{j,k} (p_j^2 - \bar{p}_k^2)^{-2} \tag{132}$$

where

$$j, j' \in \{j_1, j_2, \ldots, j_r\} \tag{133}$$

$$k, k' \in \{k_1, k_2, \ldots, k_r\}. \tag{134}$$

Similarly, we also have

$$R'(j_1, j_2, \ldots, j_r)$$
$$= \sum_{0 \leqslant k_1 < k_2 < \ldots < k_r \leqslant N} Q'(j_1, j_2, \ldots, j_r; k_1, k_2, \ldots, k_r)$$
$$\times Q''(j_1, j_2, \ldots, j_r; k_1, k_2, \ldots, k_r). \tag{135}$$

The summation can obviously be decomposed into two parts; one is extended to $k_1=0$, the other to $k_1 \geqslant 1$. The latter is just $R(j_1, j_2, \ldots, j_r)$ on account of (127) and (128). We thus have

$$\det(I+R')-\det(I+R)$$

$$= \sum_{r=1}^{N} \sum_{1\leqslant j_1<j_2<\ldots<j_r\leqslant N} \sum_{1\leqslant k_2<\ldots<k_r\leqslant N} Q'(j_1, j_2, \ldots, j_r; 0, k_2, \ldots, k_r) \times Q''(j_1, j_2, \ldots, j_r; 0, k_2, \ldots, k_r). \tag{136}$$

Using (131), we have

$$Q''(j_1, j_2, \ldots, j_r; 0, k_2, \ldots, k_r) = \prod_j C_j \prod_k \bar{p}_k \bar{C}_k \prod_{j<j'} (p_j^2-p_{j'}^2) \prod_{k>k'} (\bar{p}_k^2-\bar{p}_{k'}^2) \prod_{j,k} (p_j^2-\bar{p}_k^2)^{-1} \tag{137}$$

$$Q'(j_1, j_2, \ldots, j_r; 0, k_2, \ldots, k_r) = \prod_j C_j \prod_k \bar{p}_k \bar{C}_k \prod_{j<j'} (p_j^2-p_{j'}^2) \prod_{k>k'} (\bar{p}_k^2-\bar{p}_{k'}^2) \prod_{j,k} (p_j^2-\bar{p}_k^2)^{-1} \times \prod_j p_j^{-2} \prod_k \bar{p}_k^2 \tag{138}$$

where j and j' satisfy (133), but

$$k, k' \in \{k_2, \ldots, k_r\}. \tag{139}$$

From (120), (121) and (70), we have

$$\overline{(\rho_0)_{22}} = 1 + C(I+R)^{-1} Q \bar{p}^{-1} \bar{C}^T \tag{140}$$

and then

$$\overline{(\rho_0)_{22}} = [\det(I+R)]^{-1} \det(I+R'') \tag{141}$$

where

$$R'' = R + Q\bar{p}^{-1}\bar{C}^{\mathrm{T}} C. \tag{142}$$

It is easily seen that R'' can be expressed as

$$R'' = S'S''^{\mathrm{T}} \tag{143}$$

where

$$S'_{jk} = C_j \frac{1}{p_j^2-\bar{p}_k^2} \bar{C}_k \tag{144}$$

$$S''_{jk} = C_j \frac{p_j^2}{p_j^2-\bar{p}_k^2} \bar{C}_k. \tag{145}$$

We also have

$$\det(I+R'') = 1 + \sum_{r=1}^{N} \sum_{1\leqslant k_1<k_2<\ldots<k_r\leqslant N} \sum_{1\leqslant j_1<j_2<\ldots<j_r\leqslant N} S'(k_1, k_2, \ldots, k_r; j_1, j_2, \ldots, j_r) \times S''(k_1, k_2, \ldots, k_r; j_1, j_2, \ldots, j_r). \tag{146}$$

In analogy to (93), using (92), we have

$$S'(k_1, k_2, \ldots, k_r; j_1, j_2, \ldots, j_r)$$
$$= \prod_k C_k \prod_j \bar{C}_j \prod_{k<k'} (p_k^2 - p_{k'}^2) \prod_{j>j'} (\bar{p}_j^2 - \bar{p}_{j'}^2) \prod_{k,j} (p_k^2 - \bar{p}_j^2)^{-1} \tag{147}$$

$$S''(k_1, k_2, \ldots, kr; j_1, j_2, \ldots, j_r)$$
$$= \prod_k C_k \prod_j \bar{C}_j \prod_{k<k'} (p_k^2 - p_{k'}^2) \prod_{j>j'} (\bar{p}_j^2 - \bar{p}_{j'}^2) \prod_{k,j} (p_k^2 - \bar{p}_j^2)^{-1} \prod_k p_k^2 \tag{148}$$

where k, k' and j, j' satisfy (134) and (133), respectively. The complex conjugate of the product of (147) and (148) is obviously equal to (132). We have thus shown that

$$\det(I + R'') = \overline{\det(I + R)} \tag{149}$$

and

$$\overline{(\rho_0)_{22}} = [\det(I + R)]^{-1}\, \overline{\det(I + R)}. \tag{150}$$

It is obvious that

$$|(\rho_0)_{22}| = 1 \tag{151}$$

which can also be directly obtained from (78).

8. Explicit expression of the N-soliton solution of the DNLS equation

We obtain the explicit expression of the N-soliton solution of the DNLS equation

$$u_N = -2\frac{\det(I + R') - \det(I + R)}{[\det(I + R)]^2}\, \overline{\det(I + R)} \tag{152}$$

where $\det(I + R)$ and $\det(I + R') - \det(I + R)$ are expressed as (129) and (136), respectively.

It is convenient to rewrite (129) and (136) as

$$\det(I + R) = \sum_{(\mu)} \mathscr{D}_1(\mu) \exp\left(\sum_{j=1}^{2N} (2w_j - \ln \alpha_j)\mu_j + \sum_{1 \le j < k \le 2N} A_{jk}\mu_j\mu_k \right) \tag{153}$$

$$\det(I + R') - \det(I + R)$$
$$= \sum_{(\mu)} \mathscr{D}_2(\mu) \exp\left(\sum_{j=1}^{2N} (2v_j - \ln \alpha_j)\mu_j + \sum_{1 \le j < k \le 2N} A_{jk}\mu_j\mu_k \right) \tag{154}$$

where

$$w_j = \ln(f_j^{-1}) \qquad v_j = \ln(f_j^{-1} p_j^{-1}) \qquad \text{for } j = 1, 2, \ldots, N \tag{155}$$

$$\left.\begin{array}{ll} p_j = \bar{p}_{j-N} & \alpha_j = \bar{\alpha}_{j-N} \\ w_j = \ln(\bar{f}_{j-N}^{-1} \bar{p}_{j-N}) & v_j = \ln(\bar{f}_{j-N}^{-1} \bar{p}_{j-N}^2) \end{array}\right\} \text{ for } j = N+1, N+2, \ldots, 2N \tag{156}$$

$$\exp A_{jk} = (p_j^2 - p_k^2)^2 \qquad \text{for } j, k = 1, 2, \ldots, N \text{ or } j, k = N+1, N+2, \ldots, 2N \tag{157}$$

$$\exp A_{jk} = (p_j^2 - p_k^2)^{-2} \qquad \text{for } j = 1, 2, \ldots, N \text{ and } k = N+1, N+2, \ldots, 2N \tag{158}$$

and where

$$\mathscr{D}_1(\mu) = \begin{cases} 1 & \text{when } \sum_{j=1}^{N} \mu_j = \sum_{j=N+1}^{2N} \mu_j \\ 0 & \text{otherwise} \end{cases} \tag{159}$$

$$\mathscr{D}_2(\mu)=\begin{cases}1 & \text{when } \sum_{j=1}^{N}\mu_j=\sum_{j=N+1}^{2N}\mu_j+1\\ 0 & \text{otherwise}\end{cases} \tag{160}$$

$$\mu_j=1 \text{ or } 0 \tag{161}$$

and the summations in (153) and (154) are extended to all possible combinations of values of $\mu_1, \mu_2, \ldots, \mu_{2N}$ in (161).

Although the form of expressions in (153) and (154) is similar to that used in the direct method of Hirota (1973) for the NLS equation, so far as we know, the DNLS equation has never been solved by this method. It may be realised with reference to (152).

9. Concluding remarks

We conclude the present paper by saying a few words on the initial value problem of the DNLS equations. Since we have found explicit expressions for multisoliton solutions, the initial values corresponding to different choices of ζ_j and b_j will be completely defined. The initial value given by (42)-(45) in the paper of Ichikawa and Abe (1988) is hardly realised by choosing ζ_j and b_j even in the two-soliton case. The initial value problem of the DNLS equation seems more complicated than that of the NLS equation; we believe that the explicit expressions for multisoliton solutions will be conducive to the analysis of this problem.

We have given a systematic study of soliton solutions of the DNLS equation, from presentation of a method based on the Darboux transformation in the form of a pole expansion to an expression for explicit multisoliton solutions. This study will provide a sound basis for further research on problems relating to Alfven solitons in plasmas.

References

Anderson D and Lisak M 1983 *Phys. Rev.* A **27** 1393
Chen Z Y, Huang N N and Xiao Y 1988 *Phys. Rev.* A **38** 4355
Hirota R *J. Math. Phys.* 1973 **14** 805
Ichikawa Y H and Abe Y 1988 *Suppl. Prog. Theor. Phys.* **94** 128
Kaup D J and Newell A C 1978 *J. Math. Phys.* **19** 798
Mikhailovskii A B, Petviashvili V I and Fridmann A M 1976 *JETP Lett.* **24** 53
Mio K, Ogino T, Minami K and Takeda S 1976 *J. Phys. Soc. Japan* **41** 265
Mjølhus E 1976 *J. Plasma Phys.* **16** 321
Mjølhus E and Wyller J 1986 *Phys. Scr.* **33** 442
Wadati M, Konno K and Ischikawa Y H 1979 *J. Phys. Soc. Japan* **46** 1965